Topological Discrete Algebra, Ground State Degeneracy, and Quark Confinement in QCD

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Abstract

Based on the permutation group formalism, we present a discrete symmetry algebra in QCD. The discrete algebra is hidden symmetry in QCD, which is manifest only on a space-manifold with non-trivial topology. Quark confinement in the presence of the dynamical quarks is discussed in terms of the discrete symmetry algebra. It is shown that the quark deconfinement phase has the ground state degeneracy depending on the topology of the space, which gives a gauge-invariant distinction between the confinement and deconfinement phases. We also point out that new quantum numbers relating to the fractional quantum Hall effect exist in the deconfinement phase.

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The purpose of this paper is to present an argument for classification of phases in QCD. Classification of phases in QCD is an old but unsolved problem in quantum field theory. (For recent reviews, see [1, 2].) As is well-known, behaviors of the Wilson loop [3] (or Polyakov line [4]) and the 't Hooft loop [5, 6] are useful to classify the quark confinement and deconfinement phases in the pure Yang Mills theories, but once the dynamical quarks are included, they are no longer sufficient to distinguish them. Nevertheless, it will be shown below that there exist quantum numbers which distinguishes these two phases. In particular, they are useful to study the quark confinement in the numerical studies.

This work is motivated by recent developments on understanding of quantum phases. Many quantum phases and phase transitions can be described by the spontaneous symmetry breaking and local order parameters, however in recent years it has become increasingly clear that in a wide class of strongly correlated many-body systems, a phase transition driven by a nonthermal parameter may occur at zero temperature which can not be understood by any local order parameter. The characteristic signature of the novel phase is a finite ground-state degeneracy depending on the topology of the space, and the underling order of the novel phase is dubbed as topological order [7]. The Laughlin state for the fractional Hall effect is known to have a topological order [8]. In the present, many systems including bosonic ones and those at zero magnetic field are identified as possessing topological orders [9, 10, 11, 12, 13, 14, 15, 16, 17].

Recently, we have argued that the topological degeneracy in a topological order is due to the emergence of a discrete symmetry [18], which contains three fractional parameters: quasiparticle charge, anyon statistics, and the fractional quantum Hall conductivity. In particular, it is notable that the emergence of collective excitations having fractional quantum numbers with respect to the constitute particles in the Hamiltonian is closely related to the existence of the topological order [18, 19]. Such a charge fractionalization has an interesting analogy of the quark deconfinement, which also gives fractional charged excitations. In spite of the essential difference that the quarks are elementary particles, not collective excitations, this motivates us to study the quark confinement in the notion of the topological order.

In the following, we will show that the quark deconfinement phase in QCD indeed has a topological order. Generalizing the argument in [18], we will construct a discrete symmetry algebra in QCD, which we dub topological discrete algebra. The existence of the center of the gauge group is crucial for the construction. By the use of the topological discrete algebra,

it will be shown that the quark deconfinement phase in QCD has a ground state degeneracy depending on the topology of the system. The topological degeneracy in the thermodynamic limit is a gauge-invariant quantum number that distinguishes the deconfinement phase from the confinement one.

For definiteness we will consider the lattice QCD. The generalization to the continuum QCD is straightforward. The action of the link variable $U_{n,\mu} \in SU(3)$ is given by

$$S_{\mathrm{G}} = \sum_{\mathrm{P}} \frac{1}{g^2} \mathrm{tr}(1 - U_{\mathrm{P}}), \tag{1}$$

with the plaquette variable $U_{\mathrm{P}} = U_{n,\mu} U_{n+\hat{\mu},\nu} U^{\dagger}_{n+\hat{\nu},\mu} U^{\dagger}_{n,\nu}$, and that of the quarks $\psi^f_{\boldsymbol{n}}$ is

$$S_{\mathrm{F}} = -\frac{1}{2} \sum_f \sum_{n,\mu} \left(\bar{\psi}^f_n \gamma_\mu U_{n,\mu} \psi^f_{n+\hat{\mu}} - \bar{\psi}^f_{n+\hat{\mu}} \gamma_\mu U^{\dagger}_{n,\mu} \psi^f_n \right) - \sum_{f,n} m_f \bar{\psi}^f_n \psi^f_n, \tag{2}$$

where $n = (n_1, n_2, n_3, n_4)$ denotes the site on the lattice, $\hat{\mu}$ a unit vector in the n_μ direction, and f the indices of the flavors of the quarks. To remove the doublers, we also add the Wilson term S_{W}. The partition function $\mathcal{Z}$ is given by

$$\mathcal{Z} = \int \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-(S_{\mathrm{G}} + S_{\mathrm{F}} + S_{\mathrm{W}})}. \tag{3}$$

If necessary, other terms preserving the electric charge (improved actions for gauge field, the chemical potential terms for quarks, and so on) can be included, which does not affect the arguments in the following.

The topological discrete algebra we will consider is defined only if the topology of the space-manifold is non-trivial, so let us consider the system on a three dimensional torus T^3. (The space-time is four dimensional.) On the torus T^3, the site at $\boldsymbol{n} = (n_1, n_2, n_3)$ is identified with one at $\boldsymbol{n} + \hat{a} N_a$, where $\hat{a}$ is a unit vector in the n_a direction, and N_a is an integer ($a = 1, 2, 3$). The torus size is $N_1 N_2 N_3$. In practice, the torus is realized by imposing the periodic boundary conditions in all the spatial directions. The torus admits three independent spatial non-contractable loops C_a ($a = 1, 2, 3$) along the n_a direction. In addition, the torus has three "holes" h_a ($a = 1, 2, 3$) outside T^3. The hole h_a is encircled by the non-contractable loop C_a. In other words, each S^1 in $T^3 = S^1 \times S^1 \times S^1$ has a noncontractable loop and a hole. See Fig.1.

To define the topological discrete algebra, we introduce an external $U(1)$ electro-magnetic gauge field $e^{i\theta_{n,\mu}}$. Then consider an adiabatic insertion of the flux Φ_a through "hole" h_a

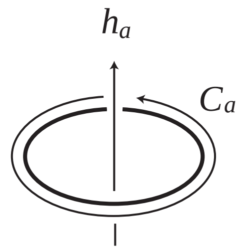

FIG. 1: Non-contractable loop C_a on S^1.

encircled by the non-contractable loop C_a. This process induces the external $U(1)$ electromagnetic gauge field $e^{i\theta_{n,\mu}}$ with $\theta_{n,\mu} = \delta_{\mu,a}\Phi_a/N_a$. Note that the $U(1)$ field strength on T^3 remains to be zero after the flux insertion.

Let us now consider the partition function $\mathcal{Z}(\Phi_a)$ with the inserted flux Φ_a. Since the upper and down quarks have $2/3$ and $-1/3$ electric charges, respectively, then by the unitary transformation,

$$\begin{aligned} &\psi_n \to e^{-i2n_a\Phi_a/3N_a}\psi_n, \quad \text{for upper quarks,} \\ &\psi_n \to e^{in_a\Phi_a/3N_a}\psi_n, \quad \text{for down quarks,} \end{aligned} \tag{4}$$

the induced $U(1)$ electromagnetic gauge field is eliminated in the action except on the links connecting the sites $\boldsymbol{n}$ with $n_a = 1$ and $n_a = N_a$. After the transformation (4), the kinetic terms of the quarks on these boundaries acquire the $U(1)$ phase as

$$\begin{aligned} &e^{i2\Phi_a/3}\bar{\psi}_n\gamma_a U_{n,a}\psi_{n+\hat{a}}, \quad \text{for upper quarks,} \\ &e^{-i\Phi_a/3}\bar{\psi}_n\gamma_a U_{n,a}\psi_{n+\hat{a}}, \quad \text{for down quarks,} \quad (n_a = N_a). \end{aligned} \tag{5}$$

The same $U(1)$ phase remains in the kinetic term in the Wilson term. If Φ_a is a unit flux 2π, these $U(1)$ phases coincide with an element of the center of $SU(3)$, $e^{-2\pi i/3}$. So performing the unitary transformation on link variables $U_{n,a}$ on the boundaries, $U_{n,a} \to e^{2\pi i/3}U_{n,a}$, $(n_a = N_a)$, one can show $\mathcal{Z}(2\pi) = \mathcal{Z}(0)$. Therefore, the spectrum of the system is invariant under the adiabatic flux insertion by 2π. The unit flux 2π is smaller than that in [18] where the unit flux is $2\pi/e$ with e the minimal charge of the constituent particles. In QCD, $e = 1/3$. The adiabatic insertion of the unit flux 2π is represented by a unitary operator U_a.

If quarks are deconfined, the physical states are classified by the representation of the permutation group for quarks. For N quarks on T^3, the permutation group consists of σ_i

$(i = 1, \cdots, N-1)$, which exchanges the ith and $(i+1)$th quarks clockwise without enclosing any other quark, and τ_i^a ($a = 1, 2, 3$, $i = 1, \cdots, N$), which represents moving the ith quark along the non-contractable loop C_a in the n_a direction. The permutation group is given by

$$\begin{aligned} \sigma_k^2 &= 1, \quad 1 \le k \le N-1, \\ (\sigma_k \sigma_{k+1})^3 &= 1, \quad 1 \le k \le N-2, \\ \sigma_k \sigma_l = \sigma_l \sigma_k, & \quad 1 \le k \le N-3, \quad |l-k| \ge 2, \\ \tau_{i+1}^a = \sigma_i \tau_i^a \sigma_i, & \quad 1 \le i \le N-1, \quad a = 1, 2, 3, \\ \tau_1^a \sigma_j = \sigma_j \tau_1^a, & \quad 2 \le j \le N, \quad a = 1, 2, 3, \\ \tau_i^a \tau_j^b = \tau_j^b \tau_i^a, & \quad i, j = 1, \cdots, N, \quad a, b = 1, 2, 3. \end{aligned} \tag{6}$$

The generators of the permutation group have non-trivial commutation relations with U_a ($a = 1, 2, 3$). If we apply τ_i^a after the flux insertion U_a, the gauge field $e^{i\theta_{n,\mu}}$ will give rise to an Aharanov-Bohm phase $e^{-i2\pi/3}$. (Both the upper and down quarks acquire the same $U(1)$ phase.) Therefore, we obtain

$$\tau_i^a U_a = e^{-2\pi i/3} U_a \tau_i^a, \tag{7}$$

where $a = 1, 2, 3$ and $i = 1, \cdots, N$. On the other hand, because τ_i^b ($b \neq a$) and σ_i do not encircle the inserted flux by U_a, they commute with U_a,

$$\tau_i^b U_a = U_a \tau_i^b, \quad \sigma_i U_a = U_a \sigma_i. \tag{8}$$

with $a \neq b$.

Using the commutation relations (7) and (8), one can verify that $U_a U_b U_a^{-1} U_b^{-1}$ ($a \neq b$) commutes with all the permutation group generators. So by Schur's lemma, for any irreducible representation of the permutation group, $U_a U_b U_a^{-1} U_b^{-1}$ is a (unimodular) c-number. Namely,

$$U_a U_b = e^{2\pi i \lambda_{a,b}} U_b U_a, \quad \lambda_{a,b} = -\lambda_{b,a}. \tag{9}$$

In addition, U_a^3 is also a unimodular integer because it also commutes with all the permutation group generators. Therefore $U_a^3 U_b = U_b U_a^3$. Comparing this with (9), we find that $\lambda_{a,b}$ should be a rational number

$$\lambda_{a,b} = \frac{k_{a,b}}{3}, \tag{10}$$

where $k_{a,b}$ is an integer and co-prime to 3. These constants $\lambda_{a,b}$ $(a,b=1,2,3)$ are new quantum numbers in the deconfinement phase.

If the system is time-reversal invariant, these new quantum numbers are shown to be zero: For the time-reversal invariant system, the time-reversal transformation of U_a is given by $TU_aT^{-1}=c_aU_a^\dagger$ with a unimodular constant c_a. Applying the time-reversal transformation T to (9) and using the anti-Hermiticity of T, we have

$$U_a^\dagger U_b^\dagger = e^{-2\pi i\lambda_{a,b}} U_b^\dagger U_a^\dagger. \tag{11}$$

The compatibility between (9) and (11) leads to $\lambda_{a,b}=0$.

In 3+1 dimensions, the only allowed statistics for particle excitations is boson or fermion, $\sigma_i=\pm\mathbf{1}$. Then the permutation group representation is uniquely determined as $\tau_i^a=T_a$, with matrices T_a satisfying

$$T_aT_b=T_bT_a. \tag{12}$$

The commutation relations (7), (8) and (9) now reduce to

$$T_aU_b=e^{-(2\pi i/3)\delta_{a,b}}U_bT_a, \quad U_aU_b=e^{2\pi i\lambda_{a,b}}U_bU_a, \tag{13}$$

with $a,b=1,2,3$. The topological discrete algebra in the quark deconfinement phase consists of the flux insertion operators U_a and the quark winding operators T_a with the commutation relations (12) and (13).

Now we show our main claim in this paper: *If the ground state of QCD has a mass gap, the quark deconfinement phase has at least* 3^3*-fold ground state degeneracy on* T^3*.* To show this, consider the following process. First, create a pair of quark and anti-quark out of a vacuum, then move the quark by T_a. After the quark returns to the original position, we pair annihilate the quark and the anti-quark. Suppose that there is a mass gap to excitations above the vacuum space and these operations do not close the mass gap, then these processes define the operation of T_a from a vacuum to a vacuum. Since T_a $(a=1,2,3)$ commutes with each other, we can take the basis of the vacuum space which diagonalizes T_1, T_2 and T_3 simultaneously, $T_a|\boldsymbol{\eta}\rangle=e^{i\eta_a}|\boldsymbol{\eta}\rangle, \boldsymbol{\eta}=(\eta_1,\eta_2,\eta_3)$. By applying U_a $(a=1,2,3)$ to this and using (13), we have

$$\begin{aligned}
T_1\left(U_1^rU_2^sU_3^t|\boldsymbol{\eta}\rangle\right)&=e^{i(\eta_1-2\pi r/3)}U_1^rU_2^sU_3^t|\boldsymbol{\eta}\rangle,\\
T_2\left(U_1^rU_2^sU_3^t|\boldsymbol{\eta}\rangle\right)&=e^{i(\eta_2-2\pi s/3)}U_1^rU_2^sU_3^t|\boldsymbol{\eta}\rangle,\\
T_3\left(U_1^rU_2^sU_3^t|\boldsymbol{\eta}\rangle\right)&=e^{i(\eta_3-2\pi t/3)}U_1^rU_2^sU_3^t|\boldsymbol{\eta}\rangle,
\end{aligned} \tag{14}$$

where r, s and t are integers. Therefore, it is found that there are 3^3 distinct sets of eigenvalues of T_a's. This implies that the ground state (vacuum) in the quark deconfinement phase has at least 3^3-fold degeneracy.

On the other hand, if quarks are confined, the topological discrete algebra becomes trivial and the ground state degeneracy obtained above disappears: In the confinement phase, the permutation group for hadrons, not for quarks, classifies the physical states. The permutation group for hadrons is also defined by (6) if σ_i and τ_i^a are interpreted as those for hadrons. On the contrary to the quark deconfinement phase, however, all the generators of the permutation group for hadrons commute with the flux insertion operators U_a $(a = 1, 2.3)$. This is because hadrons have integer electric charges. From this property, the movement τ_i^a of a hadron around the inserted flux $\Phi_a = 2\pi$ gives only the trivial Aharanov-Bohm phase, which leads to $\tau_i^a U_b = U_b \tau_i^a$, $\sigma_i U_a = U_a \sigma_i, (a, b = 1, 2, 3)$. Then by the Schur's lemma, the flux insertion operator U_a reduces to a unimodular constant for any irreducible representation of the permutation group for hadrons. In addition, since the allowed representations of the permutation group for hadrons are fermion and boson, τ_i^a for a hadron is again uniquely determined as $\tau_i^a = \tilde{T}_a$ with mutually commuting matrices $\tilde{T}_a$ $(a = 1, 2, 3)$. Because all the elements of the topological discrete algebra commute with each other, no ground state degeneracy is obtained in the confinement phase.

Note that the ground state degeneracy obtained in the quark deconfinement phase depends on the topology of the space-manifold. This is easily seen by considering the system on a 3-dimensional box with the free boundary conditions, which is homotopic to the 3-dimensional ball B^3. Because no non-contractable loop exists on this space-manifold, the permutation group consists of only the exchange operators σ_i, and does not include τ_i^a. Moreover, the operators U_a $(a = 1, 2, 3)$ can not be defined. So no topological discrete algebra is defined and no ground state degeneracy is obtained in this space-manifold. In general, if the space-manifold on which the system is defined has l independent spatial non-contractable loops, then the minimal ground state degeneracy in the deconfinement phase becomes 3^l. Our results here indicate that the deconfinement phase in QCD is topologically ordered, and the quark confinement and deconfinement transition is described properly by the concept of topological order.

In the static limit of QCD, the minimal topological degeneracy obtained above is reproduced by a conventional argument using the Wilson loop along the non-contractable loop

C_a, $W(C_a) = \mathrm{tr}\prod_{n\in C_a} U_{n,a}$. In this limit, all the quarks are infinitely heavy and decoupled from the dynamics. So the system is effectively described by the pure $SU(3)$ gauge theory. The pure $SU(3)$ gauge theory is invariant under the transformations,

$$U_{n,a} \to e^{2i\pi m/3} U_{n,a}, \quad (m = 1, 2, 3), \quad n_a \text{ fixed}, \tag{15}$$

which rotate all spacelike links in the n_a direction at a fixed n_a by an element of the center of $SU(3)$. The Wilson loop $W(C_a)$ is transformed by this center symmetry as $W(C_a) \to e^{i2m\pi/3} W(C_a)$, so the expectation value $\langle W(C_a)\rangle$ is an order parameter for the center symmetry. In the quark confinement phase, from the area law, it follows that in the temporal gauge $\langle W(C_a, \tau) W^\dagger(C_a, \tau')\rangle \sim e^{-\sigma N_a|\tau-\tau'|}$, with the imaginary times $\tau = it$ and $\tau' = it'$. Here σ is a positive constant. Thus using the cluster property

$$\langle W(C_a, \tau) W^\dagger(C_a, \tau')\rangle \overset{|\tau-\tau'|\to\infty}{\longrightarrow} |\langle W(C_a)\rangle|^2, \tag{16}$$

we have $\langle W(C_a)\rangle = 0$, so the center symmetry is not broken. On the other hand, in the quark deconfinement phase, it is possible that $\langle W(C_a)\rangle \neq 0$, and the center symmetry can be spontaneously broken. If $\langle W(C_a)\rangle \neq 0$ for all C_a's $(a = 1, 2, 3)$, we have 3^3 different set of $\langle W(C_a)\rangle$'s, which are related to each other by the center symmetry. Thus the ground state degeneracy is 3^3-fold, and it coincides with the minimal ground state degeneracy obtained from the topological discrete algebra.

The topological discrete algebra constructed above has a similarity to the 't Hooft algebra [5, 6]. For example our relation

$$T_a U_b = e^{-(2\pi i/3)\delta_{a,b}} U_b T_a \tag{17}$$

corresponds to the following relation given by the 't Hooft,

$$W(C)B(C') = B(C')W(C)e^{2\pi i n/\mathcal{N}_c}, \quad \mathcal{N}_c = 3 \tag{18}$$

where C and C' denote closed curves in 3-dimensional space, n the number of times the curve C' winds around C in a certain direction, and $B(C')$ the 't Hooft loop along C'. However, there exist essential distinctions between them. First of all, the 't Hooft algebra can be defined only when the dynamical quarks are absent, but our topological discrete algebra is defined in the presence of the dynamical quarks. Second, the topological discrete algebra in the quark deconfinement phase is different from that in the quark confinement phase, but

the 't Hooft algebra is the same in any phases. Third, new quantum numbers $\lambda_{a,b}$, which are missing in the 't Hooft algebra, exist in the topological discrete algebra.

Now we would like to address the physical meaning of $\lambda_{a,b}$. Consider the degenerate ground states $\phi_K = |\mathbf{\Phi}\rangle_K$ $(K = 1, \cdots, d)$ in the presence of inserted fluxes $\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$. They satisfy $U_a|\mathbf{\Phi}\rangle_K = e^{i\gamma_a(\mathbf{\Phi})}|\mathbf{\Phi} + \hat{a}2\pi\rangle_K$, where $\gamma_a(\Phi_a)$ is the quantum holonomy given by

$$\gamma_a(\mathbf{\Phi}) = i\int_{\Phi_a}^{\Phi_a+2\pi} d\Phi_a \langle\phi_K|\frac{\partial}{\partial\Phi_a}|\phi_K\rangle. \tag{19}$$

From $U_aU_b = e^{2\pi i\lambda_{a,b}}U_bU_a$, it follows

$$\gamma_a(\mathbf{\Phi} + \hat{b}2\pi) + \gamma_b(\mathbf{\Phi}) - \gamma_b(\mathbf{\Phi} + \hat{a}2\pi) - \gamma_a(\mathbf{\Phi}) = 2\pi\lambda_{a,b} + 2\pi M, \tag{20}$$

where M is an integer, which is independent of K because in general the degenerate ground states are related to each other by some symmetry. Then by the Stokes's theorem, the Hall conductance σ_{ab} $(a \neq b)$ [20]

$$\frac{e^2}{hd}\sum_{K=1}^{d}\int_0^{2\pi}\int_0^{2\pi}\frac{d\Phi_a d\Phi_b}{2\pi i}\left[\left\langle\frac{\partial\phi_K}{\partial\Phi_a}|\frac{\partial\phi_K}{\partial\Phi_b}\right\rangle - (\Phi_a \leftrightarrow \Phi_b)\right] \tag{21}$$

reduces to

$$\sigma_{ab} = \frac{e^2}{h}(\lambda_{a,b} + M). \tag{22}$$

This indicates clearly that a fractional $\lambda_{a,b}$ implies the fractional quantum Hall effect.

The above discussions can be generalized to the $SU(\mathcal{N}_\mathrm{c})$ QCD. For this purpose, it is convenient to introduce a fictitious $U(1)$ gauge field as an external field, instead of the electromagnetic gauge field, and assign the fictitious $U(1)$ charge $\mathcal{Q} = 1/\mathcal{N}_\mathrm{c}$ to all quarks. Then, consider an adiabatic insertion of the fictitious $U(1)$ flux by 2π through the hole h_a of T^3, which is represented by a unitary operator $\mathcal{U}_a$ $(a = 1, 2, 3)$. The quark winding operator T_a is defined in the same way as above. The topological discrete algebra in the quark deconfinement phase is given by

$$T_aT_b = T_bT_a, \quad T_a\mathcal{U}_b = e^{(2\pi i/\mathcal{N}_\mathrm{c})\delta_{a,b}}\mathcal{U}_bT_a, \quad \mathcal{U}_a\mathcal{U}_b = e^{2\pi i k_{a,b}/l_{a,b}}\mathcal{U}_b\mathcal{U}_a, \tag{23}$$

where $k_{a,b}$ and $l_{a,b}$ are co-prime integers, and $l_{a,b}$ is a divisor of $\mathcal{N}_\mathrm{c}$. Using these relations, the minimal ground state degeneracy on T^3 in the quark deconfinement phase is found to be $\mathcal{N}_\mathrm{c}^3$. On the other hand, in the quark confinement phase, the topological discrete algebra

is trivial and no topological degeneracy arises because the charge $\mathcal{Q}$ of any hadron is an integer.

To count the ground state degeneracy, we have assumed that the system has a finite gap. While color charges are screened in the presence of the gluon mass gap, our result indicates that the quark confinement is not synonymous with the color screening. In addition, the topological discrete algebra itself is valid even in the gapless system. The concept of ground state degeneracy becomes subtle in the gapless system, but the degeneracy could be identified by examining the finite scaling carefully.

To conclude, we have argued phases in QCD by using a discrete symmetry algebra which is manifest only on a space with non-trivial topology. The topological degeneracy of the ground state, which indicate the presence of a topological order, is derived and it is found that even in the presence of dynamical quarks it is a good quantum number distinguishing the quark confinement phase from the deconfinement one.

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